#1249 9 JAN 2003 Jimmy Atkins 1-9-2003

Walking the Coast of Britain

Introduction

As you undoubtedly know from past experience with SPHEX Club titles the subject of my talk has absolutely nothing to do with overseas travel, is only vaguely related to Britain, is analogous to walking and does have some application to coastlines.
It does have to do with what we know, what we think we know and what we were taught in school. At the conclusion of my prepared remarks I would ask those of you in the academic community to describe how these concepts are being taught today.
Many of you know that I am an actuary, an insurance mathematician. For many years my job has been more about management and understanding customers than anything close to real math. But I still remember how to do multiplication and long division, either with pencil and paper or a calculator. During this Walk Around Britain we'll be doing some arithmetic. Trust me I plan to keep it simple. But we will be talking about some mathematically related concepts that I hope to make real to you.

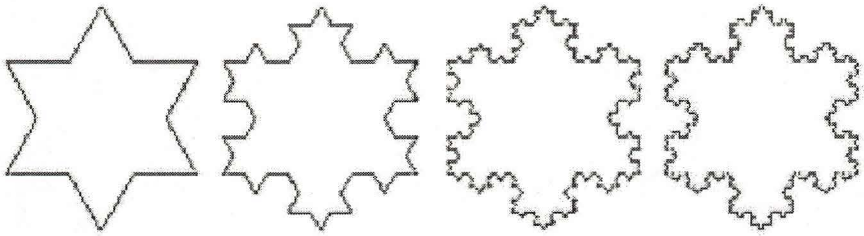
Things we know

Which are true?
All snowflakes are different.
Everything moves toward disorder.
Simple systems behave in simple ways.
Complex behavior implies complex causes.
Different systems behave differently.

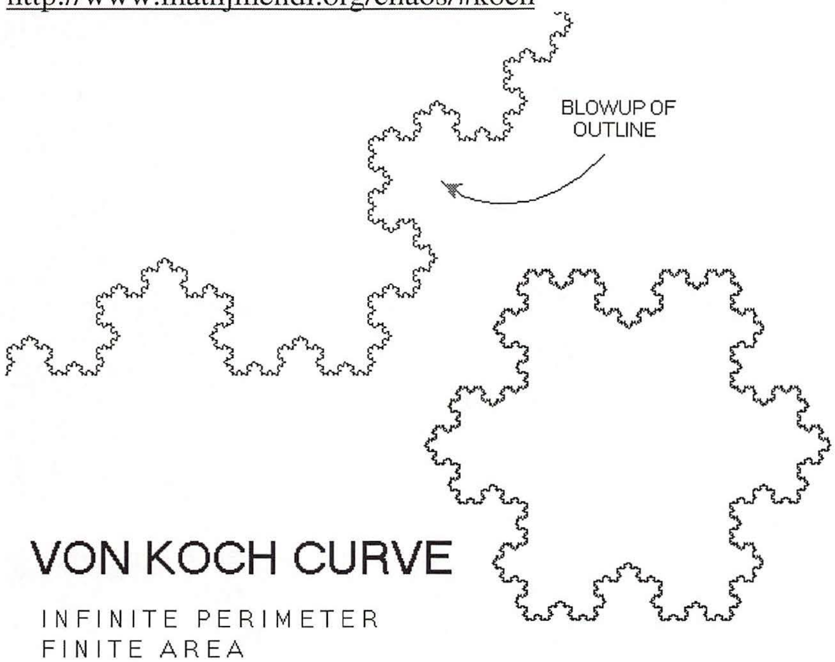
If we knew enough, we could predict the weather.

Koch Snowflake

Let's start with a simple geometric form, an equilateral triangle. Each side has length 1 so the total perimeter is 3. The area within the triangle is the (square root of 3) divided by 4. Let's just call that "A" for now.
If on each side we remove the middle 1/3 and put in two sections of 1/3 each we get a Star of David. The total length is 4/3's of 3 or 4. Each new bump is $1/9^{th}$ of the original triangle. The total area is now the original plus $1/9^{th}$ for each bump of which there were 3, 1 per side. A + A(3)(1/9) And notice now our form has 12 sides.
Do it again. The area of the addition is $1/9^{th}$ of $1/9^{th}$ for each. The total length is 4/3's of 4 (16/3). The total area = A + A(3)(1/9) + A(12)(1/9)(1/.9) or $A(1 + (3)(4^0)(1/9)^1 + (3)(4^1)(1/9)^2)$ And we now have 48 sides.



http://www.mathjmendl.org/chaos/#koch



http://www.zeuscat.com/andrew/chaos/vonkoch.html

So every time we do another iteration, we increase the perimeter by 4/3's. The area is increasing too but by a rapidly decreasing amount. If we did this an infinite number of times the total area would be 1.6A. What is the total perimeter? It is infinite. This form, with limited area has an infinitely long perimeter.

The Length of the Coast of Britain

So, How long is the coast of Britain. The answer is "it depends". It depends on what you use to measure it. We could place an imaginary rectangular fence around Britain and measure that accurately, but that would not be a good fit. From a satellite orbiting the earth we could change our scale to 500 kilometers and the coast would be about

2600 km around. As the length of the Ruler decreases, the total coastline increases.

Table 1: Length of the coast of Britain

Ruler Setting(km)	Length(km)
500.00	2600
258.82	3800
130.53	5770
65.40	8640

http://www.physics.unlv.edu/~thanki/thesis/node12.html

This is getting bigger a lot faster. As the Ruler setting drop to 1 km we have to come down to the ground to measure it. We work our way around the perimeter cove-by-cove and bay-by-bay. Then drop the Ruler to 1 centimeter. Now we are going around every pebble and every shell. How about every grain of sand? Every molecule? Every sub-atomic particle? Each time the measurement gets longer… a lot longer. Know one really knows where it stops. If these twists and turns continue forever then the length of the coastline is infinity.

Keep this technique in mind when you are trying to sell beachfront property! (1/4 acre with 50 miles of beachfront!

Well, we don't have all night to walk around the coast of Britain so let's leave this for now and consider what these shapes mean to us.

Fractal Dimension

Benoit Mandelbrot was born in Warsaw Poland in 1924. I found him on the Internet as living in Scarsdale, NY. He has been and may well still be a researcher for IBM although at age 78 he may be retired.

He has been a "Jack-of-all-Trades" in the mathematical world. As an example, he showed that in transmitting data electronically transmission errors occur in bursts and no matter how small a data segment you observe, the errors still appear in clusters and not uniformly distributed. As a result, engineers have decided to accept the appearance of transmission errors and devised strategies to fix them (oversampling in your CD player).

Another of his important contributions is the notion of Fractional Dimension.

You will recall from high school geometry that a line has but one dimension, a plane has two, and a cube or other three-dimensional object has, of course, three dimensions.

A line, a plane, and a cube are simple but abstract forms. Where do you ever see one exactly in real life? Real life is made up of objects that are a little crooked, a little bumpy.

Consider a piece of aluminum foil. (Now we all know this qualifies as a three-dimensional object but I want to stretch your imagination a bit. Imagine it is a piece of a plane, a two-dimensional object. When I wrinkle it up a little bit it moves away from this idealized form and takes on the characteristics of a three dimensional object. Maybe this form is dimension 2.1. If I ball it up I can easily throw it to you. It looks like a three-dimensional object but we know it's really just a two-dimensional object folded and twisted onto itself. This could have a dimension of 2.9.

As a quick example of how to calculate a fractional dimension compare a straight line to a line with a lot of zigs and zags. A straight line, 10 Post-It note segments long, can be covered in just 10 Post-It notes. Ten notes to cover a one-dimensional length of ten is 1 dimension. The zigzag line takes more notes. Maybe it takes 12 notes to cover the 10-note length. That would be approximately dimension 1.2. If you use a smaller sized note, it takes more to cover the crooked line. If it takes a two-dimensional object to cover it, it's two-dimensional. The closer the area of the ever shrinking Post-It Notes is to a single two-dimensional object the closer the fractional dimension is to 2. The smaller it is the closer the fractional dimension is to 1.

Fractional dimension is a measure of how bumpy or un-even an object is. From a distance a mountain looks like an inclined plane. Things should just slide right off it. Up close we see boulders, crevasses, plants. A computer graphics artist working on a film can adjust the fractional dimension of his image to produce a realistic landscape.

Knowing the fractional dimension of the rubber on you car tires and the surface of the road gives us a good way to forecast the gripping power of your car on the highway. Other applications include anticipating the transmission of water, gas or oil moving through the ground, predicting the strength of a metal, the sureness of an electrical connection and the probability of success when I try to use Superglue on that porcelain lid to the sugar bowl I dropped Monday night.

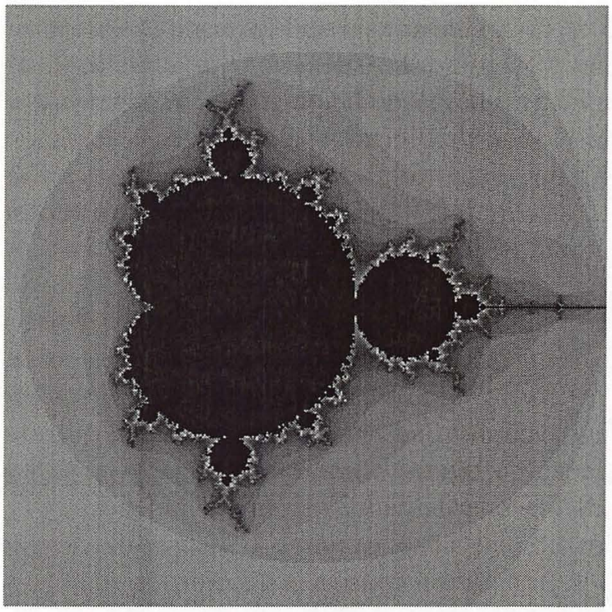
The Mandelbrot Set

In 1979 Mandelbrot started toying with an idea at IBM. He was thinking about applying a recursive formula to the expression $z = z^2 + c$ where z is a pair of coordinates on a complex plane and c is a complex constant.

Let's back up a minute and recall what a complex number is. You remember that Natural Numbers are 1,2,3, etc. and Whole numbers are 0, 1, 2, etc. Integers include the negative numbers -1, -2, -3, etc. Rational numbers are numbers that can be expressed as a ratio of integers like ½ or ¾'s. Of course you recall square roots. The square root of 9 is 3. The square root of 25 is 5. The square root of 2 is 1.4-something. It is known as an irrational number and together with the rational numbers we get the Real Number set.

But what is the square root of –9? It is not –3. Actually –3 is also a square root of 9 because negative three times negative three equals a positive nine. Well, mathematicians just defined the square root of –1 as ***i***. So the square root of –9 is the square root of –1 times the square root of 9 or 3***i*** or –3***i***.

A complex number is a number with it's imaginary component usually written as x + y***i***. If y is zero this is the same as the real number x. Mandelbrot set up a chart with the real number component on the horizontal axis and the complex component on the vertical axis. He divided it into a large grid of points. Each point was a different value for "c". The "z" variable always starts at 0. He then applied his formula $z = z^2 + c$ to itself for each of those points over and over again and observed if the number moved to +/- infinity or not. Points that moved toward infinity are out. Points that don't are in. 'In' points are colored black. 'Out' points are left blank. He turned on the computer and then waited to see what the 'In' and 'Out' points would look like on a printout. It formed an object with a strange shape. And the paper apparently had smudges where the printer was bad. On closer inspection and a more refined plot he saw these were not smudges, and that the main object had a lot of warts on it. It looked like this (his original was not in color.)



He looked closely at this object, directing the computer to work on just a small area. What he saw was that this basic form repeats itself at ever-smaller scales. Not exactly, but close enough for you to recognize the similarity. And that no matter how close you get this object never simplifies.

Mandelbrot called this the Mandelbrot Set and coined the name "Fractal" for an object that has completely irregular boundaries. The Koch curve we saw at the beginning is a simple fractal.

You could write the computer code to do these calculations yourself. It's not that much code. Even so the Mandelbrot Set has been called the most complex object ever discovered.

Let's take a trip inside the Mandelbrot set.

http://www.oz.net/~alden/animfrac/animfrac4a.html

Fractals have the property of being self-similar regardless of scale. That is, as you zoom in on a part of it you see patterns repeating themselves at ever-smaller scales. Let's go out on the Internet and dive into the Mandelbrot set.

http://www.fractal-dome.de/ezsinhz1.shtml

The Mandelbrot Set is the collection of "c" values that do not go to infinity. If you hold "c" constant and let "z" vary over the complex plane you get the Julia set, named for Gaston Julia, an early 20th century mathematician who devised the method for producing a Julia Set. Julia set images are a lot wilder for carefully chosen values of "c".

For a complete explanation of Julia Set arithmetic see the web page http://www.lifesmith.com/makjulia.html

One key to coloring these pictures is to count how many interations are required to send a particular point out of the set. A point that moves out in 10 iterations gets a different color than one that moves out after 200 iterations. You can then map any color palette to the number of iterations. If your computer only has 256 distinct colors it makes sense to assign a different color to each value and to assume that after 256 iterations, if the point is still in the set, it is going to stay in and get the color associated with 256. By rearranging the color palette you can produce different colored images emphasizing one part of the structure or another. It's like putting salt on your food. Do what you like.

Let's look at the following Julia set.

http://www.lifesmith.com/magseres2a.html

The function is the simple $f(z) = z^2 + c$ but where $c = .109950511 + .63352439\boldsymbol{i}$.

We are diving into the heart of this Julia set with greater and greater magnification. Observe that the structure is similar all the way in. And it never gets simple. The last magnification is 1.1×10^{46}. If the last picture was two inches square and you could print the whole thing on one piece of paper it would be pretty big. How big? If the paper were just 1/1000" thick is would go around the earth… sixty-four billion, billion, billion times (64,000,000,000,000,000,000,000,000,000 times). That would extend the roll of paper 100 times farther than the Andromeda galaxy!

That's big.

Art

Besides this interesting phenomenon of an infinitely deep structure what fascinates me with fractals is the art they generate. To find more links enter "fractal" on any web search engine.

http://www.fractalsgalore.co.uk

Phase Space

Ok, let make a mental shift away from the surreal view of fractal art and go back to a simple mathematical tool called a Phase space diagram. It's a way to map changes in more that one variable at a time. For example, consider a pendulum. You could plot velocity on the X-axis and angle from vertical on the Y-axis. In a frictionless environment this would form a loop and the pendulum swings back and forth forever. Subject to the drag of friction the pendulum grinds to a halt and the phase space diagram, tracing velocity and angle through time spirals into 0 velocity, 0 angle.

Lorenz

Edward Lorenz and Long-Term Meteorology

> In the early sixties, a certain meteorologist named Edward Lorenz experimented with computer simulations of weather on a relatively primitive "Royal McBee" computer. His program used twelve recursive equations to simulate rudimentary aspects of weather; he entered several variables into his program each time he ran it, and watched to see what types of weather patterns such initial conditions would generate. He could print out graphs of fluctuating temperatures or other conditions, and his program captured the fancy of his fellow meteorologists.
> http://www.zeuscat.com/andrew/chaos/lorenz.html

Lorenz's weather model had twelve independent variables. Even though Lorenz did not use this tool at first, we can see something similar to what is program produced by looking at this two dimensional representation of a three dimensional object.

http://www.exploratorium.edu/complexity/java/lorenz.html

> One day, Lorenz tried to recreate an interesting weather pattern, one he had seen previously, by re-entering the values the computer had previously calculated and

reported. However, when he ran the program again, his results were different from the initial run. Lorenz suspected a bug; blown diode? burned-out vacuum tube? power surge? *cosmic rays*? After checking the two plots, however, he realized his "error"; on his previous computer printout, the one he had used to enter the initial conditions into the computer for the second trial run, the figures were printed with three significant digits. In the program, all values were calculated to *six* significant digits. Lorenz had assumed that the difference, only one part in a thousand, would be inconsequential; however, due to the recursive nature of the equations, little errors would first cause tiny errors, which would then affect the resulting next calculation a bit more, which would affect the output of the next run even more. The final result of a long string of recursive calculations would lead to a weather pattern totally different from the expected values. http://www.zeuscat.com/andrew/chaos/lorenz.html

Lorenz simplified his model to only these three formulas

dx/dt = delta * (y - x)

dy/dt = r * x - y - x * z

dz/dt = x * y - b * z

We can see these three variables changing as time moves forward. This

He saw that with even an ever-so-small deviation in the initial settings for x, y, or z the system would get off track fairly quickly.

He traces the path of a point in the x,y,z, three-dimensional, graph and observes 1) the movement through phase-space never retraces the same point exactly, 2) he can never be sure where the system will go after only a few interations, 3) while the behavior is unpredictable, it is not unconstrained. It is neither random, nor predictable.

Let’s perform an experiment.

http://www.exploratorium.edu/complexity/java/lorenz.html

This model plots the three-dimension phase-space diagram into two dimensions. While it appears the lines intersect, in three-dimensional space they do not. Notice if we start two lines as close together as possible they still diverge.

So we can tell what the weather is likely to be for a short while although we can’t be sure. We can be pretty sure that a long-term forecast is just as likely wrong as right. But because the results seem to circle these two key points, these two Strange Attractors, heating up to an oven-like temperature is highly unlikely if not impossible nor is 40 days and 40 nights of rain in our future. The weather will be like something we have recently experienced, we just can’t say what.

Nature, with apparent disorder, is constrained. And drawn to a "Strange Attractor"

> Lorenz labeled these systems that exhibited sensitive dependence on initial conditions as having the "butterfly effect": this unique name came from the proposition that a butterfly flapping its wings in Hong Kong can effect the course of a tornado in Texas. http://www.wfu.edu/~petrejh4/HISTORYchaos.htm

The website academicpress.com defines chaos as "the dynamical evolution that is aperiodic and sensitively dependent on initial conditions." That is, it moves is a way that never repeats itself and has regions of sensitivity so that you can never have a measuring tool sensitive enough to be sure of the future of the system from those "alleged" starting points.

This Lorenz butterfly is not random. Given a staring point it will always draw the same pattern. But it is unpredictable because in real life, just like with the mouse button here, you cannot get exactly the same starting point. Even so there is order here. It does not go everywhere.

Locally unpredictable, globally stable.

Population

How does a biologist predict population growth of deer in Lynchburg?

Logistics formula

Consider the logistics equation f(x) = (r)(x)(1-x). X represents the number of deer. The more you have the more you get (birthrates). The (1-x) factor represents restrictions of the food supply. The higher the population the more likely they are to starve (or me hit by a car). The "r" parameter is a net driver of growth. The initial x can be less than 1 but more than zero. "r" can range from more than 1 to less than 4.

For low values of r the population stabilizes quickly.

For higher values of "r" the population oscillates between two values. Then, suddenly 4, then 8, then it goes completely out of control with the population bouncing all over the place! And then, just as unexpectedly, it momentarily settles down again before once again entering a seemingly random pattern.

Plotting the ultimate population values for different levels of "r" produces and image like this:
http://www.zeuscat.com/andrew/chaos/bifurcations.html

You can see the structure giving way to chaos the dropping back to a brief area of control. Notice the calm area looks like the original calm area, only smaller. (How fractal-like! Could there be a connection?)

This simple model produces what appears to be random behavior. Or that it is natural for some systems to oscillate between 2 or 4 or more values. What does that imply for human efforts to control disease? Public health programs to control measles, rubella, or other communicable diseases can be successful even if there is a short-term adverse experience. By studying measles data you can see that they follow a strange attractor, like Lorenz's weather model. It is just natural for communicable diseases to ebb and flow.

So, if this, the simplest of systems can exhibit chaotic behavior, shouldn't everyone learning any form of science learn this and know that in the real world, complexity from even simple systems is the norm and simplicity is the exception?

Feigenbaum

In 1975 Mitchell Feigenbaum was an eccentric researcher at Los Alamos. He was supposed to be working on the problem of turbulence, that nasty thing that drains energy out of systems, creates drag on airplanes and seems to crop up in real life all too frequently. He noticed the Logistics Formula was exhibiting something like turbulence. He started playing around with the Logistics Formula, trying to see exactly where the solution went from one value to two and then from two to four. His equipment was dreadfully slow so he starting looking at the emerging results and thought he saw a pattern. Sure enough the bifurcations were coming a geometrically accelerating rate. The ratio of the point it went from 1 to 2 to the point it went from 2 to 4 was 4.669. The ratio of the point it went from 2 to 4 to the point it went from 4 to 8 was also 4.669 and so on. That was interesting.

He later repeated this task for a completely different formula involving the trigonometric sin function [$f(x) = r \sin \pi x$]. Again he found a geometric constant: 4.669. Whoa. It was exactly the same. Using more sophisticated computers he determined this number was 4.6692016090. It is a universal constant, like Pi.

As he told his parents, "This is going to make me famous."

http://sprott.physics.wisc.edu/fractals/chaos/logistic.gif

Applications

So chaos is real. Chaos is universal. And chaos is everywhere now that we know to look for it. In 1978 Voyager 2 gave us a close up look at Jupiter. One of the expectations was that we would finally know what that Great Red Spot of Jupiter was. Over the years various theories had been suggested and discarded.

- A lava flow?
- New Moon rising up from the clouds?
- Solid body floating on a liquid surface?
- Column of Gas?
- Voyager 2 showed Jupiter to be totally or almost totally liquid. The boundary of the Red Spot has eddies, currents, etc. It's like a never-ending hurricane. It is stable chaos. In the early 1980's physics graduate student Phillip Marcus stepped away from his traditional education. He thought of the Great Red Spot as analogous to our Gulf Stream. He devised a computer model that showed regardless of the initial conditions the red spot would form a stable zone is a sea of chaotic blue.

Chaos and Science

Classical physics taught me that a pendulum of a given length swings at the same rate regardless of how wide it swings. That's not quite right.

In the real world, pendulums stop. A Golf ball dropped from the leaning tower of Pisa will hit the ground before a ping-pong ball dropped at the same time.

But to be fair, the tools to explore chaos are recent inventions. The computer is the tool of the chaos researcher and graphics are the way we see the results. The computer let's us see where to go next. It is a tool that reveals new ideas to us. It provides evidence and gives the mathematicians a lot more to try to prove.

> Our daily encounter with Chaos/Complexity is seen in traffic flow, weather changes, population dynamics, organizational behavior, shifts in public opinion, urban development and decay, cardiological arrhythmias, epidemics. It might be found in the operation of the communications and computer technologies on which we rely, the combustion processes in our automobiles, cell differentiation, immunology, decision making, the fracture structures, and turbulence.
> http://www.wfu.edu/~petrejh4/HISTORYchaos.htm

A modern scientist has to start with an established body of knowledge. They can't recreate all knowledge from first principles. And so they tend to not question the established dogma. It takes someone brave to step away from the established tradition or apply the techniques of one discipline to problems of another but that's when breakthroughs come. Physicist David Ruelle said it's the non-specialists who find the new things.

Traditional science has discounted some results as "experimental error". In fact, sometimes this was chaos.

Today, the real scientists are not just those who study particle physics in super-colliders. Today real scientists can study clouds.

Chaos and Biology

Chaos

Chaos is found inside us as well.

Blood pumping through the heart is subject to turbulence. Turbulence around artificial heart components results in clotting and strokes. Application of chaos mathematics to heart models has enabled better equipment that has a chance of working.

When the heart goes into a state of fibrillation it's in a steady but chaotic state. It stops working and it takes a massive jolt of electricity to push it back to its normal rhythm. Chaos is death.

The brain works in a chaotic state. Steady rhythms are signs of Parkinson's disease. Equilibrium = flat brainwaves = death. Chaos is life.

Fractals

The Fractal Fern is a classic shape. It follows a simple fractal formula but it looks just like the real thing. It is evidence that the DNA coding for a fern is a compact fractal.

- The human circulatory system has a fractal quality as bigger arteries branch into ever smaller ones to the point that now cell is more than a cell or two from a blood vessel but the circulatory system and the blood it carries consumes only about 5% of the total body volume
- Lung tissues are like the folded up piece of aluminum foil. A tremendous surface area in a small volume
- Bronchial tubes in those lungs branching ever smaller

We all know that DNA encodes our genes and out genes describe our physical structure. I recently meet a geneticist from Louisiana State University. She said that most of our DNA is "just stuff" that can identify us but does not guide our development. DNA cannot describe every cell in our body. But it can, with a simple, fractal representation, describe even the most complex structures of our bodies.

Chaos is everywhere. We just have to open our eyes to see it. Fractals are everywhere in nature. We are fractal beings.

Things we know

Things we know	
Simple systems behave in simple ways.	No
Complex behavior implies complex causes.	No
Different systems behave differently.	Not always
If we knew enough, we could predict the weather.	No chance
Everything moves toward disorder.	Not everything
All snowflakes are different.	Absolutely

Conclusion

Twentieth century science will be known for three things: relativity, quantum mechanics and chaos.

Chaos has reversed a trend in science toward reductionism, the belief that by understanding the smallest parts of a system you could understand the system as a whole.

Chaos and fractals are with us, are in us, are us. We just have to have our eyes open to see them.