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THE GOLDEN BLOCKHEAD,

Leonardo Pisano, Fiboacci Numbers and the Golden Mean

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I must begin by giving you the good news. Being one of the few remaining people not offended or defensive about being labeled a Liberal Democrat, I have spent the last six months deciding which one of my socially, economically or political liberal soapboxes I would climb tonight. I considered topics like the history of the concept of the Common Good or the over-emphasized role and rights of Capital versus the other stakeholders of Corporations or the ultimate pitfalls of Ayn Rand's "I" Society.

But I decided to leave the path of causes and, at the risk of a boring evening, take the path of frivolity.

I decided instead to return to my roots as I did in previous papers on Andrew Carnegie and First Colony. What some of you may not realize that in my early life, I was an Actuary. Some have theorized that an Actuary is an individual who is numerically competent but lacking the personality to be an accountant. Sound exciting yet?

I could tell actuarial jokes like "...did you hear about the Actuary who drowned walking across a river with an average depth of 4 feet."

But all my life I have both worked with and been entertained by numbers. I heard Herb Moore talk to this group about childhood memories----well many of my most fond childhood memories center around math. My father worked in the steel mill for 45 years but he was quite a numbers man. He believed the highest calling in life, one step above President, was to be a teacher and most of all, a high school math teacher.

One of the things I love about math, especially recreational math and puzzles, are the surprises it presents. I compare it to bird watching. One wanders through the forest of numbers and comes upon a rare find....I thought would fall over laughing when I compared math to bird watching, something she likes to do. And so tonight I would like to share some of these surprises with you. They have little practical applicability to us in this room, they have no moral or ethical implication, they are not going to change the world but I hope you will find them, as I do, fun.

For years, I have been fascinated by the name Fibonacci (again, MaryJane wanted to know why I was going to talk about a liar) and what little I knew about his work. So I decided he and his work would be my topic tonight. Simple enough. Finite universe of material, rather understandable and totally non-controversial.

Would you believe none of these are true? There is more material than I could ever digest or cover here tonight. There is a Fibonacci Society and a quarterly Fibonacci Journal presenting papers on related topics. As for it being understandable, well, like many things, there is a basic level which is understandable but there are aspects and implications far beyond which we, limited by time and my knowledge, will travel tonight but one could quickly get in over one's head.

While it is certainly not emotionally controversial as some of my "liberal" causes might be, there are certain controversies swirling around even this subject.

Lets begin our journey.

In 250 BC, one of the greatest scholarly works in History was written. It was the *Elements* by Euclid. In spite of Euclid being a best selling author(only the bible sold more books than the *Elements* prior to the twentieth century), his life is veiled in such obscurity that even his birthplace is unknown. Given the contents of the *Elements*, it is likely that Euclid studied under students of Plato.

The *Elements*, a 13 volume work on geometry and number theory, is so colossal in scope that we often forget that Euclid was also the author of at least 12 other books covering topics from music through mechanics to optics. Only four survive today with one, Optics, containing some of the earliest studies and theories on perspective.

Few would disagree that the *Elements* was the greatest and most influential mathematics textbook ever written. A story has it that when Lincoln wanted to understand the true meaning of the word Proof, he started to study the *Elements* in a cabin in Kentucky. Likewise, Bertrand Russell, the great British philosopher, mathematician and logician, describes his first encounter with the *Elements* (at age 11) as "one of the greatest events of my life, as dazzling as first love",

Even Sherlock Holmes declared in a *Study in Scarlet* claimed that his conclusions achieved by deduction were "as infallible as so many propositions of Euclid".

In the *Elements*, Euclid attempted to encompass most of the mathematical knowledge of his time. Books I to VI deal with the plane geometry that we learn in school today and which has become synonymous with his name. Books VII to X deal with number theory and the foundation of arithmetic. Book XI provides the basis for solid geometry. Book XII proves the theorem for the area of a circle and Book XIII demonstrates the constructions of the five Platonic Solids. (See Figure I)

It is in Book VI where, the Golden Ratio, aka the Extreme and Mean Ratio first appears. He uses this Golden Ratio in the construction of the pentagon and in the construction of the icosahedron and dodecahedron.

What is the Golden Ratio?

Johannes Kepler, the great astronomer, mathematician and philosopher said:

> *Geometry has two great treasures: one is the Theorem of Pythagoras;*
> *The other, the division of a line into extreme and mean ratio,*
> *The first we may compare to a measure of gold,*
> *The other we may name a precious jewel*

So it is "a precious jewel" but what is it an emerald or sapphire or diamond?

Lets look at Figure II. In Figure I1, the Line AB is divided by Point C. Point C is located such that the ratio of the larger segment to the smaller segment is equal to the ratio of the whole line to the larger segment or

A **C** **B**

AC/CB = AB/AC = The Golden Ratio

How is this ratio related to the pentagon?

In any regular planar figure, (equal sides and interior angles), the sum of all interior angles is 180°(n-2). For a pentagon, this total would be 540° (3*180°) or 108° for each angle. Imagine we draw two adjacent diagonals in the Pentagon (Figure III) forming 3 isosceles triangles. Since the two base angles of an isosceles triangle are equal, the base angles on the two triangles on the sides are 36° or {half of (180° – 108°)}. Therefore the angles of the middle triangle are 36°, 72° and 72°. If we bisect one of the 72° angles, we obtain a smaller triangle DBC with the same angles(36°,72°,72°) as the large one ADB. Euclid then shows that point C divides the line AB in the Golden Ratio. Furthermore, the ratio of AD to DB is also equal to the Golden Ratio.

Bored yet? I promise that proof is the last of those tonight.

In other words, in a regular pentagon, the ratio of the diagonal to a side is equal to *phi*, the Golden Ratio. This fact provides a simple means of constructing a regular pentagon which was the reason this ratio was so important to Euclid.

The triangle with a ratio of side to base of *phi* is known as the Golden Triangle. The side triangles with a ratio of side to base of (1/*phi*) are known as Golden Gnomons. Figure III also demonstrates a unique property of the Golden Triangle and Golden Gnomons--- they can be dissected into smaller triangles that are also Golden Triangle and Golden Gnomons.

Let us now look at the Golden Rectangle in Figure IV. The sides in a Golden Rectangle are in Golden Ratio to each other. Suppose we cut off a square from the rectangle. We will be left with another Golden Rectangle. The dimensions of the "daughter" rectangle will smaller than those of the "parent" rectangle by a factor of *phi*. If we cut another "daughter" from the "daughter" we will be left with another golden rectangle with dimensions smaller again by a factor of *phi*. Continuing this ad infinitum, we will create a series of golden Rectangles each successive one deflated by a factor of *phi*. The Golden Rectangle is the only known rectangle with the property that cutting a square from it produces a similar rectangle.

Draw two diagonals of any mother-daughter rectangles in Figure IV and they will intersect at the same point. The series of continuously diminishing rectangles converges to that never-reachable point. Another surprise

The association of the Golden Ratio with the pentagon and the platonic solids was important and interesting given the Pythagorean fascination with the pentagon coupled with Plato's interest in regular solids and also his belief that the latter represented fundamental cosmic entities. It prompted generations of mathematicians to labor on theorems concerning *phi*. Yet this interest would not have reached the level it did and has were it not for some unique algebraic properties and its relation to and occurrence in nature, architecture, music etc.

Returning to Figure II, if we have the smaller segment, CB, be 1 Unit and the length of the longer segment, AC, be x, then:

$$x/1 = (x+1)/x,$$

Multiplying both sides by x, we get:

$$x^2 - x - 1 = 0$$

The solution to this quadratic equation is:

$$x = (1 \pm \sqrt{5})/2$$

Surprise!

The positive solution is 1.6180339887.............The Golden Ratio.

This clearly is an irrational number being one half the sum of one and the square root of 5.

Among its interesting properties, the square of the Golden Ratio yields 2.6180339887... and its reciprocal yields 0.6180339887... The Golden Ratio has the unique property that to get its square, add 1 and to get its reciprocal, subtract 1.

Another surprise.

To show mathematicians are not without a light side, Paul Bruckerman published an amusing poem, "Constantly Mean" with the first verse:

The Golden Mean is quite absurd,
It's not your ordinary surd,
If you invert it(this is fun!),
You'll get itself, reduced by one,
But if increased by unity,
This yields its square, take it from me.

Now that we have an algebraic expression for the Ratio, it can be calculated to a high precision. In what I have read, the leader is the calculation to 10 million decimal places.

We now fast forward in our surprise machine to 1202, when many scholars would say the story of numbers in the western culture began. In that year a book titled *Liber Abaci* or Book of the Abacus appeared in Pisa (*Leaning Tower*) Italy. The fifteen-chapter book was entirely handwritten---printing would not be invented for another 300 years. The author was Leonardo Pisano who was only 27 years old but a very lucky man. His book would receive the endorsement of Frederick II---no author could have done better than that.

. His father's first name was Bonacio. Bonacio means "simpleton" and Fibonacci is a contraction of son-of-Bonacio. Pisano was known most of his life as Fibonacci, the name by which he is known today and ……Fibonacci means "blockhead".

Surprise---The Golden Blockhead!

But he was much more than a blockhead. Both he and his father represented Pisa as Consul in a number of different cities. Fibonacci was inspired to write *Liber Abaci* on a visit to Bugia, a thriving Algerian city where his father was serving as Consul. While there, an Arab mathematician taught him the Hindu-Arabic numbering system that Arab mathematicians had introduced to the West during the Crusades.

When Fibonacci saw all the calculations which this system made possible---calculations which were impossible with Roman numerals---(See new Figure)he set about learning everything he could about it. To study with the leading Arab mathematicians living around the Mediterranean, he set off on a trip that took him to Egypt, Syria, Greece, Sicily and Provence.

The result was a book, extraordinary by any standard. It made people aware of a whole new world where numbers could be substituted for the Roman, Greek and Hebrew systems, all of which used letters for calculating.(Figure IIc) The book rapidly attracted a following in Italy and throughout Europe.

Liber Abaci is far more than a primer for working with the new numerals. Fibonacci begins with instructions on how to determine from the number of digits in a numeral whether it is a unit, or a multiple of ten, or a multiple of 100, and so on. Later chapters exhibit a higher level of sophistication. There we find calculations using whole numbers and fractions, rules of proportion, extraction of square roots and roots of higher orders, and even solutions for linear and quadratic equations.

The book commanded an enthusiastic following because he filled it with practical applications. For example, he described and illustrated many innovations which numbers made possible in commercial bookkeeping such as figuring profit margins, money changing, conversion of weights and measures and---although still widely prohibited---he even included the calculation of interest payments.

Let us take a brief historical detour.

Though Frederick, who ruled from 1211 to 1250, exhibited cruelty and an obsession with power, he was genuinely interested in science, the arts and the philosophy of government. In Sicily, he destroyed all the feudal castles, taxed the clergy, and banned them from civil office. He also set up an expert bureaucracy, abolished internal tolls, removed all regulations inhibiting imports and shut down most state monopolies.

Frederick tolerated no rivals. Unlike his grandfather, Frederick Barbarossa, who was humbled by the Pope at the Battle of Legnano in 1176, this Frederick reveled in his endless battles with the papacy. His intransigence brought him not just one but two excommunications. On the second occasion, Pope Gregory IX called for Frederick to be disposed, characterizing him as a "…heretic, rake, and anti-Christ....". Frederick responded with a savage attack on the papal territory and he captured a large delegation of prelates on the way to Rome to join the synod called to remove him from power.

Frederick surrounded himself with the leading intellectuals of his age, inviting many of them to join him in Palermo. Frederick was fascinated with *Liber Abaci*. Some time in the 1220s, while on a visit to Pisa, he invited Fibonacci to appear before him. In the course of the interview, Fibonacci solved problems in algebra and cubic equations put to him by one of Frederick's many scientists-in-residence. Fibonacci subsequently wrote a book prompted by this meeting, *Liber Quadratorum*, or *The Book of Squares*, which he dedicated to the Emperor.

But to get back on track.

Fibonacci is best known for a short passage in *Liber Abaci* that led to something of a mathematical miracle. The passage presents the popular rabbit problem which asks:

> How many rabbits will be born in the course of a year from an original pair of rabbits assuming each month each pair produces another pair and that rabbits begin to breed when the are two months old.

The problem can be solved by drawing a picture beginning with the original pair of rabbits in the first month r.(Figure V) By the second month, they will have produced a pair of babies. By the third month, they will have produced a second pair of babies and the first pair will have matured. One month later, they will have another new pair of babies the second pair will have matured and the first pair will have babies of their own. Fibonacci discovered that the original pair of rabbits would spawn a total of 233 pair of offspring in the course of a year. If the rabbits kept going for a hundred months, the total number of pairs would be 354,224,848,179,261,915,075.(354 quintillion 224 quadrillion etc)

The drawing can continue as shown in Figure V but it rapidly becomes unmanageable. A tally of the number of pairs(Figure V) of rabbits each month however reveals a fascinating pattern by which the problem can be solved. But it is not the answer, 377, but rather the pattern that distinguishes the problem. The sequence of the number of rabbits begins with 1 and each number which follows is the sum of the previous two numbers. Because this work of Fibonacci is the earliest known recording of this infinite sequence, Edouardo Lucas, the French mathematician, labeled it the Fibonacci sequence made up of Fibonacci numbers.

The rabbit problem and its solution is contrived---rabbits don't breed that way but the Fibonacci sequence does itself occur naturally in many other contexts. It appears repeatedly in nature in objectively identifiable forms and in mathematical and geometrical arenas. It has also been identified as appearing in such diverse human creative endeavors as art, architecture, music, poetry and even stock market analysis. These are the areas where much controversy has occurred.

First, I will address its appearance in nature, then in man-created art forms and last the "surprises" in mathematics.

One if the most reliable places to look for Fibonacci numbers in nature is in the growth pattern of plants.(Figure VII) Plants tend to grow in spirals as the reach for the sky. The tip of a branch moves in a circular motion reaching for nutrients, it also gets longer because as it is growing the tip spirals in space. Thus growth spirals are characterized by both a circular motion and elongation.(Figure VII)

As a branch grows it generates leaves at regular intervals but not typically after each complete circle of its spiral. One explanation for this is that leaves generated after a complete circle would grow directly over each other sheltering the lower ones from the very elements they need to grow. It appears that leaves are most often generated after 2/5 of a circle of growth or 3/5 or 3/8 or 8/13 etc. Curiously, these phyllotactic ratios, as they are called, usually have Fibonacci numbers in both the numerator and denominator.

Figure VII demonstrates a Phyllotactic ratio of 3/8. This ratio is determined by locating two stems directly above each other, counting how many circle turns the branch went through to do that and how many stems it created.

The spirals that characterize pinecones are an even clearer example of Fibonacci ratios. The bracts on the surface of a pinecone spiral around the cone much as leaves do around a stem. On close examination, two sets of spirals can usually be found; one going diagonally from lower left to upper right and the other crossing it diagonally from lower right to upper left. One spiral rises gradually while the other rises more steeply.(Figure VII)

Studies have shown that there is a 99 % likelihood that the number of spirals on a pinecone will be Fibonacci numbers, typically adjacent numbers.

Hexagonally shaped scales cover the outer surface of a pineapple. Each of these hexagons is on three different spirals aligned with opposite sides of the hexagon.(Figure VII) If we count the number of gradual, medium and steep spirals, they will almost always be adjacent Fibonacci numbers, usually 8, 13 and 21. These are marked on the pineapple being passed around. A study in 1977 of 2000 pineapples revealed no deviations from the Fibonacci pattern.

Sunflowers display Fibonacci numbers in their own unique way. There are two distinct spirals of seeds, one going clockwise and one counterclockwise. The usual number of spirals in a sunflower head is 34 going one way and 55 going another way. There have been large sunflowers with as many as 233 spirals in one direction. While deviations are found, sunflower spirals are overwhelmingly Fibonaccian(to coin a new word).

Many flowers exhibit Fibonacci numbers in their buds, seeds and petals. Most daisies have either 13, 21 or 34 petals. Any one playing "…loves me….loves me not…" will pull off a Fibonacci number of petals to arrive at the answer. Two other examples are delphiniums which usually have 8 petals and Black Eyed Susans which have 21 petals.

While petals and other manifestations of phyllotaxis do not follow an absolute law, they do, in the words of geometrician H. S. M. Coxeter, exhibit a "…fascinatingly prevalent tendency.." to be Fibonaccian.

The family tree of a male Bee also exhibits a Fibonacci sequence. A male bee develops from an unfertilized egg or to put it simply, a male bee has only a mother, not a father. On the other hand, a female bee develops from a fertilized egg and thus has a mother and a father. The total number of ancestors of the male bee is always a Fibonacci number.

One of the most intriguing and beautiful occurrences of the Fibonacci numbers in nature is the remarkable spiral which occurs rather often in animal growth. This spiral can be constructed as shown in Figure VI.

Construction could, of course, continue as far as space permitted, with squares of the needed dimension being added successively. Then quarter circle arcs can be drawn connecting opposite corners of the squares (using the sides as the radii of the arcs) in such a way that the arcs connect sequentially.

What develops is a beautiful spiral, known both as an equiangular or logarithmic spiral. It is called equiangular because all radii from its center intersect the spiral itself at exactly the same angle (Figure VIII). The building blocks of this particular spiral are squares whose dimensions are successive terms in the Fibonacci sequence. Visually it can be described as a long, slow spiral. It is quite different from a more "evenly spaced" spiral, as represented, for example, by the groove on a phonograph record and known as an Archimedian spiral.

Perhaps the clearest and loveliest example of an equiangular spiral is the shell of the chambered nautilus. As a nautilus grows, it is necessary for the chamber in which it lives to become larger to accommodate its increased size and yet stay the same shape to accommodate the contour of its body. An examination of the cross section of the shell shows how this is done (Figure VIII). As the shell gets larger, the radii also get larger, but the angles of the intersection of the radii and the outer shell remain the same. Therefore, the chambers will be similarly shaped but larger. Each time this happens, the living nautilus moves into new, familiar, but roomier quarters, where it lives comfortably until the process needs to be repeated.

Many equiangular spirals can be found in the animal kingdom; examples include the horns of wild sheep,, spider webs. Parrot beaks, cat and canary claws, elephant tusks, the graph of bacterial growth, and the path of an insect approaching light. Most of these occurrences seem to be linked to the basic characteristic of the spiral retaining the same shape while getting larger. Although not all equiangular spirals exhibit the Fibonacci proportions, most do, and others (such as animal horns) appear to but are difficult to verify because they do not lie in a plane.

Fibonacci numbers also play a role in regular pentagons which occur often in nature. More flowers bloom in pentagons than in any other shape. The sand dollar, the sea urchin, and the starfish(Figure IX) are examples of pentagons found in the sea. The insides of fruits and vegetables are often pentagonal in shape.

The proportions of a pentagon are also related to the proportions between adjacent Fibonacci numbers. For example, if the measure of the side of a pentagon is a Fibonacci number, the measure of its diagonal is the next Fibonacci number in the sequence. Furthermore, the diagonals of any such pentagon separate each other into two adjacent Fibonacci numbers. These principles are illustrated in Figure X with the Fibonacci numbers 55, 89 and 144.

Occurrences of the Fibonacci sequence in nature---like the sequence itself—seem never to end. Pine needles tend to grow in clusters of 2, 3, or 5, depending on the species. The number of chambers in many plant pods is a Fibonacci number. The mathematics of snowflake construction yields a Fibonacci ratio. Much the same way that bracts spiral around a pinecone—in Fibonacci configurations—the units of protein spiral around the flagella (tails) of single-celled organisms as they swim.

The search for additional natural occurrences of the Fibonacci sequence can be a source of endless curiosity and infinite satisfaction.

Now to its occurrence in man-created forms of Art. Note the definition of Art is beyond tonight's scope.

Evidence of the use of the golden rectangle in architecture can be found throughout history and all over the world; sometimes the evidence is concealed, other times more obvious. It is buried deeply inside the tomb of Ramses IV, built in Egypt 1,400 years after the Great Pyramid. The burial chamber consists of three rectangles, a small one inside a middle-sized one inside a large one. The small one is a double square, the middle one is a golden rectangle, and the large one is a double golden rectangle (see figure 3-8). The Greeks appear to have been strongly influenced by golden proportions, consciously or unconsciously, since the classical age of Greek culture beginning in the fifth century BC. Many scholars would say that geometric analysis of Greek statues and artifacts, such as vases, urns, and so on, reveals extensive use of the golden ratio. Use of the proportion can be found abundantly in the work of Phidias, considered the greatest of Greek sculptors and for whom that the golden ratio was named "*phi*". Supposedly this is discernable in the bands of sculpture designed by Phidias that run above the columns of the Parthenon.

Much of Renaissance art and architecture was inspired by the Greek sense of beauty and proportion. It is not surprising, then, that so many buildings, statues, and tombs of that prolific era are characterized by golden ratios—doorways, floor plans, windows, gates, overall dimensions, and so on.

Today, the presence of golden proportions in contemporary architecture is not an accident. The famous twentieth-century architect Le Corbusier was deeply committed to the use of golden proportions in his work. His influence can be found in buildings ranging from private villas in France to the headquarters of the United Nations in New York City. He is not alone; other architects use the golden proportion to greater or lesser extents.

The "divine proportion," as Luca Pacioli called it, also figures prominently in the works of many great painters. The proportion may exist in various forms—simple exterior dimensions, for example, or an underlying grid. A collection of such paintings may include representatives from virtually all traditions and styles, ranging from the old masters to modern artists. The late nineteenth-century French impressionist artist Seurat is said to have "attached every canvas with the golden rectangle." Durer, Mondrian, and Bellows are other artists who are thought to have consciously incorporated the golden ratio into their work.

The Golden Ratio is thought by some to have also been "sighted" in music. Perhaps the clearest link between Fibonacci numbers and music can be found on the keyboard of a piano. An octave on a keyboard is made up of 8 white keys and 5 black keys. The black keys are positioned in groups of 2 and 3. There are 13 keys altogether in one octave, an analysis of which involves each of the first six Fibonacci numbers.

Keyboard aside, those 13 notes belong to what is known as the chromatic scale, the most complete scale to have developed in Western music. Its principal predecessor was the 8-note diatonic scale, better known as the octave, which was preceded by the 5-note pentatonic scale. The pentatonic scale was used in early European music and is the basis today of much music education for young children. Any 5 consecutive black keys on the keyboard constitute such a scale. A number of well-known folk tunes can be played using just these keys; examples include "Mary Had a Little Lamb," "Ring Around the Rosy," "Go Tell It on the Mountain," and "Amazing Grace." Although other scales have existed, the pentatonic (5), diatonic (8), and chromatic (13) scales dominate the development of Western music.

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The musical intervals considered y many to be the most pleasing to the ear are the major sixth and minor sixth. A major sixth, for example, consists of C, vibrating at about 264 vibrations per second, and A, vibrating at about 440 vibrations per second. The ratio of 264 to 440 reduces to 3/5, a Fibonacci ratio. An example of a minor sixth would be E (about 330 vibrations per second), and C (about 528 vibrations per second). The ratio, 330 to 528, reduces to 5/8, the next Fibonacci ratio. The vibrations of any sixth interval reduce to a similar ratio. It has been suggested that the Fibonacci numbers are part of a natural harmony that not only looks good to the eye but also sounds good to the ear.

Perhaps this is why composers have consciously or unconsciously incorporated Fibonacci numbers and proportions into their work. Scholarly analyses of a wide range of music including Gregorian chants, Bach fugues, and Bartok sonatas reveal this to be true.

We now move from the naïve to the sublime sightings

Far more sophisticated accusation are the sightings in Poetry with Virgil's Aeneid, the epic Roman poem, being most frequently mentioned. Professor G.E. Duckworth of Princeton University, who claims to have identified frequent use of the Fibonacci sequence to create golden proportions and rhythms, analyzed this literary masterpiece in the 1960s. Some feel other poets of Virgil's time also used these proportions to structure their poems. Ancient poetry was intended to be heard because only the elite were educated enough to read it. It may not be surprising then that the harmony and mathematical balance of music should be found in poetry as well since both strive to be pleasing to the ear.

At the same time, most serious scholars would take issue with Duckworth's work feeling that Virgil certainly did not consciously use the Fibonacci sequence structure in the Aeneid or any of his other work.

Another relatively recent but well-known development involves attempts by Ralph Elliott to use the Fibonacci sequence and the Golden Ratio in the analysis of stock market patterns. An accountant by profession,(too much personality to be an Actuary), Elliott analyzed in great detail the rallies and plunges of the Dow Jones Industrial Average. During his lifetime, Elliott witnessed the roaring bull market of the 1920's followed by the Great Depression. His detailed analyses led him to conclude that market fluctuations were not random. In particular, he noted; "the stock market is a creation of man and therefore reflects human idiosyncrasy." Elliott's main observation was that, ultimately stock market patterns reflect cycles of human optimism and pessimism.

In, 1935, Elliott wrote a treatise entitled *The Wave Principle.* In it he claimed to have identified characteristics which "furnish a principle that determines the trend and gives clear warning of reversal." This treatise eventually developed into a book with the same title, which was published in 1938.

Elliott's basic idea was relatively simple. He claimed that market variations can be characterized by a fundamental pattern consisting of five waves during an upward ("optimistic") trend (marked by numbers in Figure XI) and three waves during a downward ("pessimistic") trend (marked by letters in Figure XI). Note that 5, 3, 8, the total number of waves, are all Fibonacci numbers. Elliott further asserted that an examination of the fluctuation on shorter and shorter time scales reveals that the same pattern repeats itself (Figure XI), with all the numbers of the constituent wavelets corresponding to higher Fibonacci numbers. Identifying 144 as "the highest number of practical value," the breakdown of a complete market cycle, according to Elliott, might look like an upward trend consisting of five major waves, twenty-one intermediate waves, and eighty-nine minor waves followed by a generally downward phase with three major, thirteen intermediate, and fifty-five minor waves.

Some recent books attempt to apply Elliott's general ideas to actual trading strategies. They use the Golden Ratio to calculate the extreme points of maximum and minimum that can be expected (although not necessarily reached) in market prices at the end of upward or downward trends. Even more sophisticated algorithms include a logarithmic spiral plotted on top of the daily market fluctuations, in an attempt to represent a relationship between price and time.

All of these forecasting efforts assume that the Fibonacci sequence and the Golden Ratio somehow provide the keys to the operation of mass psychology. However, this "wave" approach does suffer from some shortcomings. The Elliott "wave" usually is subjected to various and often rather arbitrary stretching, squeezing and other alterations by hand to make it "forecast" the real-world market. Investors know, however, that even with the intelligent application of all the bells and whistles of modern portfolio theory, which is supposed to maximize the returns for a decided-on-level of risk, there are many smart but broke investors.

Elliott's original wave principle represented a bold if somewhat naïve attempt to identify a pattern in what appears otherwise to be a rather random process. There has been much very "scholarly" research on this application of Fibonacci numbers but this research has yielded few practical results. Recently, this research has led to examining relationships between Fibonacci numbers and fractals. This research has yielded some practical benefits and holds promise for more but you will be happy to know that this is beyond tonight's scope mainly because it is beyond my understanding and knowledge.

It is this almost "cultish" devotion to finding the shadow of the Golden Ratio in so many human creative endeavors which has produced much controversy. There is a large group of scholars including Martin Gardner, the popular recreational mathematician, who have written extensively about the fallacies of so many of the purported "sightings" especially in the areas of art, architecture, music and poetry.

While they're clearly have been excesses in this area, there is no doubt about its existence and importance in mathematics and nature.

One more area which deserves some discussion is the spiritual element of the Golden Mean.

The concept of the Golden Ratio continued to attract attention with the Publication in Venice in 1509 of the *Divine Proportion* by Luca Pacioli. In this book, he raves about the properties of the Golden Ratio. He analyses in succession what he calls the thirteen different effects of the *Divine Proportion* and attaches to each adjectives like "essential", "singular", "supreme and so on. Pacioli stops at thirteen concluding, "for the sake of salvation, the list must end because thirteen men were present at the table at the Last Supper"

To ensure the book's success, Pacioli secured the services of the dream illustrator---Leonardo da Vinci. Da Vinci provided sixty illustrations of geometric solids depicted in both skeletal and solid forms. Da Vinci and Pacioli continued to be friends and collaborators for most of their lives.

In the *Divine Proportions*, Pacioli lists five reasons why he believes that the appropriate name for the Golden Ratio should be the Divine Proportion.

1. "There is one only and not more". He compares this uniqueness to the fact that unity "is the supreme epithet of God himself".

2. Pacioli finds a similarity between the fact that the Golden Ratio involves precisely three lengths and the "threeness" of the Holy Trinity, Father, Son and Holy Spirit.

3. To Pacioli, the incomprehensibility of God and the fact that the Golden Ratio is an irrational number are equivalent. "Just like God cannot be properly defined, nor can be understood through words, likewise our proposition cannot ever be designated by intelligible numbers, nor can it be expressed by any rational quantity but always concealed and secret and called irrational by the mathematician."

4. He compares the omnipresence and invariability of God to the self-similarity associated with the Golden Ratio---that its value is always the same and does not depend on the length of the line being dissected.

5. The fifth reason reveals a very Platonic view of existence and again, quite beyond tonight. Pacioli states that just as God conferred being to the entire cosmos through the fifth essence represented by the dodecahedron, so does the Golden Ratio confer being on the dodecahedron since one cannot construct the dodecahedron without the Golden Ratio

The publication of this book provided new and additional respect for the Golden Ratio. The infusion of theological and philosophical meaning into the name "Divine Proportion" singled out the Golden Ratio as a mathematical topic into which an increasingly eclectic "bevy" of intellectuals could delve. Finally, the concept of the Golden Ratio became available to artists in theoretical treatises which were not overly mathematical.

Now for what I consider the real fun and where the fun surprises occur---the mathematics of Fibonacci numbers.

Again, mathematician's humor, J. A. Lindon wrote the following:

> *Each wife of Fibonacci*
> *Eating nothing that wasn't starchy*
> *Weighed as much as the two before her*
> *His fifth was some signora!*

Appendix I shows what Fibonacci numbers are. That is we start with 0 and 1 as F_0 and F_1 and define $F_n = F_{n-2} + F_{n-1}$. For example, the 10^{th} number in the sequence is 55 or the sum of the 8^{th}, 21 and the 9^{th}, 34. The second column in Appendix 1 shows the cumulative sum of the Fibonacci numbers. The third Column shows the ratio of the n and n-1 numbers.

Eureka! Bells! Whistles! Alarms! Fireworks!

We have just discovered the biggest surprise of the night. We see the ratio of two adjacent Fibonacci numbers oscillates around and asymptotically approaches as a limit the Golden Ratio. In English, that says that as the Fibonacci numbers increase, the ratio of adjacent numbers gets closer and closer to the Golden Ratio.

Note the Lucas numbers. The relationship between the numbers is similar to Fibonacci numbers with each number being the sum of the two preceding numbers. The difference is that the Lucas numbers begin with one and three while the Fibonacci numbers begin with one and one.

Realize what we have discovered.

A Geometric relationship first discovered by Euclid what now been found to have an Algebraic "brother" i.e. the ratio of two adjacent Fibonacci numbers approaches, as a limit, the Golden Mean---or maybe it is the Divine Proportion of Pacioli. We could now pursue the philosophic and mathematical implications of the relationship but I am going to declare that to be work for a future paper and use the remaining time to have some fun. We are going to use the remaining time to search for and discover numerical surprises, related to the Fibonacci numbers, numbers which pop out from behind every tree.

SURPRISE I

No two consecutive Fibonacci numbers have any common factors.

(Factors are that set of prime numbers which when multiplied together yield the number)

Number	Factors
2	2
3	3
5	5
8	2 x 2 x 2
13	13
21	3 x 7
34	2 x 17
55	5 x 11
89	89
144	2 x 2 x 2 x 3 x 2
233	233
377	13 x 29
etc.	

SURPRISE II

The twelfth Fibonacci number is the square of 12 i.e. 144. That is the only perfect square in the infinite series. The only perfect cube is 8.

SURPRISE III

The sum of any 10 consecutive Fibonacci numbers is always evenly divisible by 11.

$8+13+21+34+55+89+144+233+377+610 = 1584 \qquad 1584 \div 11 = 144$

SURPRISE IV

As a corollary to Surprise III, the sum of any 10 consecutive Fibonacci numbers is equal to the seventh number in the series times 11.

$8+13+21+34+55+89+144+233+377+610 = 1584 \qquad 144 \times 11 = 1584$

SURPRISE V

For any four consecutive Fibonacci numbers, F_n, F_{n+1}, F_{n+2}, F_{n+3}

$$(F_{n+2})^2 - (F_{n+1})^2 = F_n \times F_{n+3}$$

For 5, 8, 13, 21; $169 - 64 = 5 \times 21 = 105$

SURPRISE VI

The famous mathematician Joseph LaGrange discovered that the unit digit of the Fibonacci numbers is cyclic with a periodicity of 60. That means the unit digit of the 1st, 61st, 121st, etc Fibonacci numbers is the same. The same is true for the 2nd, 62nd and 122nd numbers. Similarly, the last two digits are cyclic with a periodicity of 300, the last three digits are cyclic with a periodicity of 1,500, 15,000 for four digits, 150,000 for five digits, 1,500,000 for six digits, or generally, the last n digits are cyclic with a periodicity of 15 times ten to the power of one less than the number of digits etc.

n	F_n	F_{n+60}	F_{n+120}
0	0	154800875592**0**	..........184**0**
1	1	250473078196**1**	..........192**1**
2	1	405273953788**1**	..........376**1**
3	2	655747031984**2**	..........568**2**
4	3	1061020985772**3**	..........944**3**
5	5	1716768017756**5**	..........512**5**

SURPRISE VII

Every 3rd Fibonacci number is divisible by 2
Every 4th Fibonacci number is divisible by 3
Every 5th Fibonacci number is divisible by 5
Every 6th Fibonacci number is divisible by 8
Every 7th Fibonacci number is divisible by 13
Every 8th Fibonacci number is divisible by 21
....and etc

SURPRISE VIII

The eleventh Fibonacci number (89) is singularly remarkable. Its reciprocal, 1/89, can be generated by adding the Fibonacci numbers … albeit very carefully.

```
  0.0112358
          13
           21
            34
             55
              89
              144
               233
                377
                 610
                  987
1/89 = 0.01123595505617787     etc.
```

SURPRISE IX

Twice any Fibonacci number minus the next Fibonacci number equals the second number preceding the original one.

1
1
2
3
5
8   8 x 2 = 16   16 – 13 = 3
13
21

34
55
89 $89 \times 2 = 178$ $178 - 144 = 34$
144
233
377

SURPRISE X

The product of any two alternating Fibonacci numbers differs from the square of the middle number by 1. The difference is alternately plus or minus as the series continues.

1
1
2
3

5 8 13	$8^2 = 64$ $13 \times 5 = 65$ $65 - 64 = 1$
21 34 55	$34^2 = 1156$, $21 \times 55 = 1155$ $1155 - 1156 = -1$

89

SURPRISE XI

The sum of the squares of any 2 consecutive Fibonacci numbers F_n and F_{n+1} is F_{2n+1}

n	F_n	$(F_n)^2$
5	5	25
6	8	64
11	89	

$25 + 64 = 89$

SURPRISE XII

If Fibonacci numbers are squared and the adjacent squares are added together, a sequence of alternate Fibonacci numbers emerges

n	F_n	$(F_n)^2$	(Sum)	(Diff)	(Diff)	(Diff)	(Diff)
1	1	1					
			2				
2	1	1		3			
			5		5		
3	2	4		8		8	
			13		13		13
4	3	9		21		21	
			34		34		34
5	5	25		55		55	
			89		89		89
6	8	64		144		144	
			233		233		233
7	13	169		377		377	
			610		610		
8	21	441		987			
			1597				
9	34	1156					

SURPRISE XIII

The difference of the squares of alternate Fibonacci numbers is always a Fibonacci number:

Number	Square	Difference
1	1	
1	1	
2	4	4 – 1 = 3
3	9	9 – 1 = 8
5	25	25 – 4 = 21
8	64	64 – 9 = 55
13	169	169 – 25 = 144
21	441	441 – 64 = 3773
34	1156	1156 – 169 = 987

***SURPRISE XIV REPEAT**

For any four consecutive Fibonacci numbers, the difference of the squares of the middle two numbers equals the product of the smallest and largest numbers.

8
13 $13^2 = 169$
21 $21^2 = 441$ $441 - 169 = 272 = 34 \times 8$
34

SURPRISE XV

The sum of the squares of n Fibonacci numbers is $F_n \times F_{n+1}$ e.g. for n = 7

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 = 13 \times 21 = 273$$

SURPRISE XVI

For any three Fibonacci numbers, the sum of the cubes of the two greatest minus the cube of the smallest equals a Fibonacci number

F_n	$(F_n)^3$	
5	125	
8	512	
13	2197	$2197 + 512 - 125 = 2584 = F_{18}$

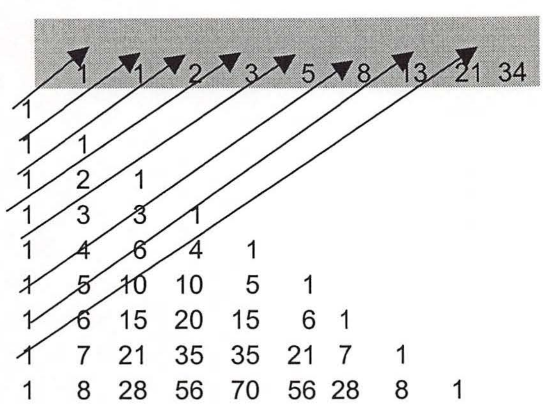
*SURPRISE XVII

The Fibonacci numbers can be found in the well-known Pascal triangle as sums of the diagonals. This pattern is much easier to identify in the Chinese adaptation of the Pascal triangle.

Pascal Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Chinese Adaptation



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The sum of any number of Fibonacci numbers beginning with the first 1 is equal to 1 less than the second number beyond the last one added. This is also true for any Fibonacci type series with the subtractive number being the second number in the series e.g. for a Lucas number, 3 would be subtracted from the second number beyond the last one added. This relationship has often been used by magicians and entertainers to demonstrate lightning quick adding skills.

See Table of Fibonacci numbers

*SURPRISE XIX

With the exception of 3, every Fibonacci number which is prime has a prime subscript. For example, 233 is F_{13} and its subscript, 13, is prime. Put another way, if a subscript is composite (non-prime), so is the number. Unfortunately, the converse is not true, i.e. not all Fibonacci numbers with a prime subscript are prime. The first counter example is F_{19}. 19 is prime but $F_{19} = 4{,}181$ is 17 times 113.

The last item that I am going to leave with you is not a surprise but rather a question which no mathematician has yet been able to answer.

SURPRISE QUESTION XX

Is there an infinite number of primes in the series? We know there is an infinite number of Fibonacci numbers but we have not been able to prove there is an infinite number of prime Fibonacci numbers. Have at it and you too can be a "famous mathematician",

With that, we will conclude our walk through the forest of surprises. I hope you have had as much fun as I have---thank you for your patience.

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Markowski, G. "Misconceptions about the Golden Ratio," *College Mathematics Journal.* 23(1992): 2-19

This paper depended heavily on the above sources, especially the Gardner, Garland and Livio books.

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n	F_n	F_{n+60}	F_{n+120}
0	0	1548008755920	1840
1	1	2504730781961	1921
2	1	4052739537881	3761
3	2	6557470319842	5682
4	3	10610209857723	9443
5	5	17167680177565	5125

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Every 4th Fibonacci number is divisible by 3
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         21
          34
           55
            89
            144
             233
              377
               610
                987
1/89 = 0.01123595505617787   etc.
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			2				
2	1	1		3			
			5		5		
3	2	4		8		8	
			13		13		13
4	3	9		21		21	
			34		34		34
5	5	25		55		55	
			89		89		89
6	8	64		144		144	
			233		233		233
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1 2 1
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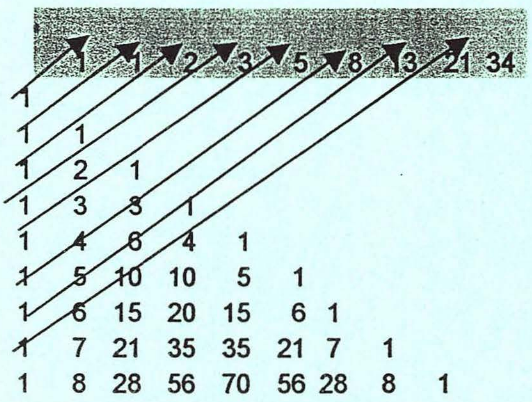
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Chinese Adaptation



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QUESTION XX

Is there an infinite number of primes in the series?

With that, we will conclude our walk through the forest of surprises. I hope you have had as much fun as I have.

APPENDIX I

N	Fibonacci Numbers F^n	Sum F^n	Ratio F^n/F^{n-1}	Lucas Numbers L^n	Sum L^n	Ratio L^n/L^{n-1}
0	0	-		-	0	
1	1	1		1	1	
2	1	2	1.0000000000	3	4	3.0000000000
3	2	4	2.0000000000	4	8	1.3333333333
4	3	7	1.5000000000	7	15	1.7500000000
5	5	12	1.6666666667	11	26	1.5714285714
6	8	20	1.6000000000	18	44	1.6363636364
7	13	33	1.6250000000	29	73	1.6111111111
8	21	54	1.6153846154	47	120	1.6206896552
9	34	88	1.6190476190	76	196	1.6170212766
10	55	143	1.6176470588	123	319	1.6184210526
11	89	232	1.6181818182	199	518	1.6178861789
12	144	376	1.6179775281	322	840	1.6180904523
13	233	609	1.6180555556	521	1,361	1.6180124224
14	377	986	1.6180257511	843	2,204	1.6180422265
15	610	1,596	1.6180371353	1,364	3,568	1.6180308422
16	987	2,583	1.6180327869	2,207	5,775	1.6180351906
17	1,597	4,180	1.6180344478	3,571	9,346	1.6180335297
18	2,584	6,764	1.6180338134	5,778	15,124	1.6180341641
19	4,181	10,945	1.6180340557	9,349	24,473	1.6180339218
20	6,765	17,710	1.6180339632	15,127	39,600	1.6180340143
21	10,946	28,656	1.6180339985	24,476	64,076	1.6180339790
22	17,711	46,367	1.6180339850	39,603	103,679	1.6180339925
23	28,657	75,024	1.6180339902	64,079	167,758	1.6180339873
24	46,368	121,392	1.6180339882	103,682	271,440	1.6180339893
25	75,025	196,417	1.6180339890	167,761	439,201	1.6180339885
26	121,393	317,810	1.6180339887	271,443	710,644	1.6180339888
27	196,418	514,228	1.6180339888	439,204	1,149,848	1.6180339887
28	317,811	832,039	1.6180339887	710,647	1,860,495	1.6180339888
29	514,229	1,346,268	1.6180339888	1,149,851	3,010,346	1.6180339887
30	832,040	2,178,308	1.6180339887	1,860,498	4,870,844	1.6180339888
31	1,346,269	3,524,577	1.6180339888	3,010,349	7,881,193	1.6180339887
32	2,178,309	5,702,886	1.6180339887	4,870,847	12,752,040	1.6180339888
33	3,524,578	9,227,464	1.6180339887	7,881,196	20,633,236	1.6180339887
34	5,702,887	14,930,351	1.6180339887	12,752,043	33,385,279	1.6180339887
35	9,227,465	24,157,816	1.6180339887	20,633,239	54,018,518	1.6180339887
36	14,930,352	39,088,168	1.6180339887	33,385,282	87,403,800	1.6180339887
37	24,157,817	63,245,985	1.6180339887	54,018,521	141,422,321	1.6180339887
38	39,088,169	102,334,154	1.6180339887	87,403,803	228,826,124	1.6180339887
39	63,245,986	165,580,140	1.6180339887	141,422,324	370,248,448	1.6180339887
40	102,334,155	267,914,295	1.6180339887	228,826,127	599,074,575	1.6180339887
41	165,580,141	433,494,436	1.6180339887	370,248,451	969,323,026	1.6180339887
42	267,914,296	701,408,732	1.6180339887	599,074,578	1,568,397,604	1.6180339887
43	433,494,437	1,134,903,169	1.6180339887	969,323,029	2,537,720,633	1.6180339887
44	701,408,733	1,836,311,902	1.6180339887	1,568,397,607	4,106,118,240	1.6180339887
45	1,134,903,170	2,971,215,072	1.6180339887	2,537,720,636	6,643,838,876	1.6180339887
46	1,836,311,903	4,807,526,975	1.6180339887	4,106,118,243	10,749,957,119	1.6180339887
47	2,971,215,073	7,778,742,048	1.6180339887	6,643,838,879	17,393,795,998	1.6180339887
48	4,807,526,976	12,586,269,024	1.6180339887	10,749,957,122	28,143,753,120	1.6180339887
49	7,778,742,049	20,365,011,073	1.6180339887	17,393,796,001	45,537,549,121	1.6180339887
50	12,586,269,025	32,951,280,098	1.6180339887	28,143,753,123	73,681,302,244	1.6180339887

F_{100} =354,224,848,179, 261,915,075

Golden Mean= 1..618033988749894848204586834365638117720309179805762862135448622705260462818902449707

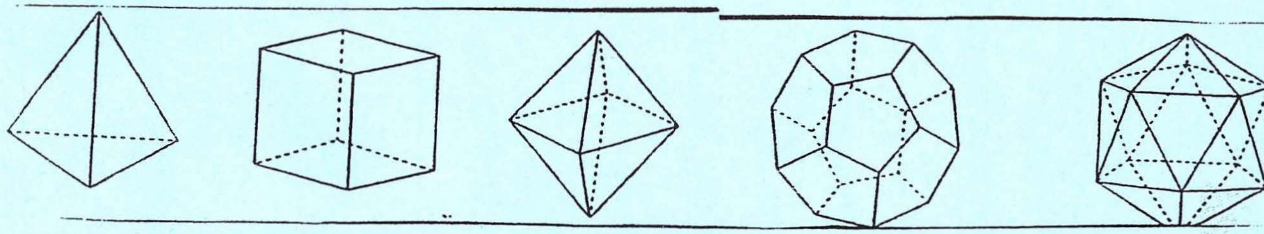
(85 Decimal Places)

Passed out blue, green, white -
Not afraid to be a liberal [illegible] -

Ring the bell -

Application -
Beale Treasure ?

FIGURE I

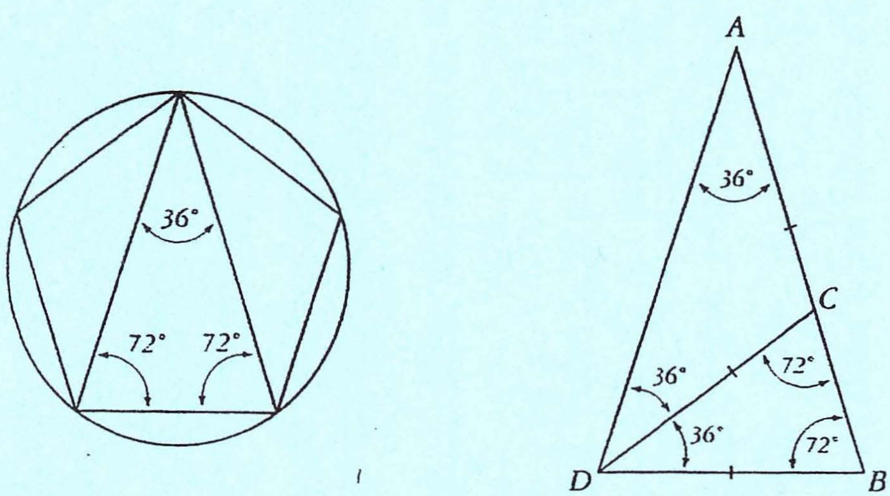


Tetrahedron Cube Octahedron Dodecahedron Icosahedron

FIGURE II

AC/CB = AB/AC = The Golden Ratio

FIGURE III



(a) (b)

FIGURE IV

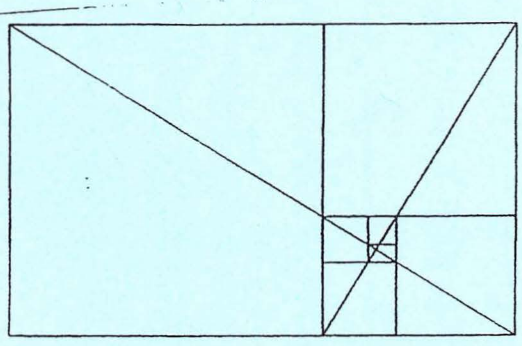


FIGURE IIa

Define the length of the smaller segment, CB, as 1 Unit and the length of the longer segment, AC, as x, then:

$$x/1 = (x+1)/x,$$

Multiplying both sides by x, we get: $x^2 - x - 1 = 0$

The solution to this quadratic equation is: $x = (1 \pm \sqrt{5})/2$

Surprise....

The positive solution to this equation is 1.6180339887.............The Golden Ratio.

FIGURE IIb

The Golden Mean is quite absurd,
It's not your ordinary surd,
If you invert it(this is fun!),
You'll get itself, reduced by one,
But if increased by unity,
This yields its square, take it from me.

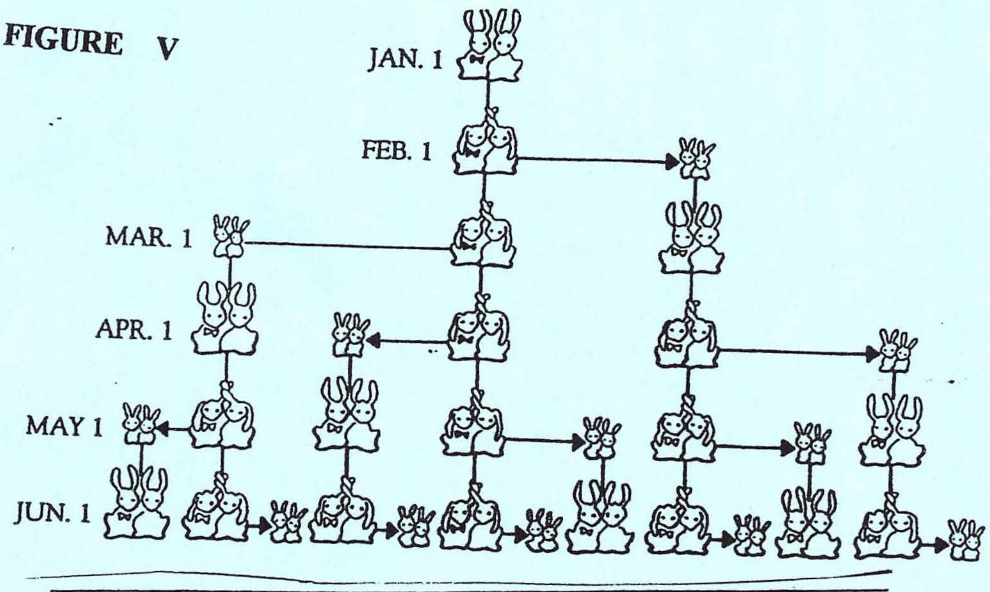
FIGURE IIc

Arabic Numbers vs. Roman Numerals

3,786 + 3,843 = 7,629

MMMDCCLXXXVI + MMMDCCCXLIII = MMMMMMMDCXXIX

FIGURE V



Month	No. of Pairs of Babies	No. of Pairs of Adults	Total No. of Pairs
Jan. 1	0	1	1
Feb. 1	1	1	2
Mar. 1	1	2	3
Apr. 1	2	3	5
May 1	3	5	8
Jun. 1	5	8	13
Jul. 1	8	13	21
Aug. 1	13	21	34
Sep. 1	21	34	55
Oct. 1	34	55	89
Nov. 1	55	89	144
Dec. 1	89	144	233
Jan. 1	144	233	377

FIGURE VI

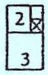
Construction of an equiangular spiral.

1. Begin with a 1-unit square.

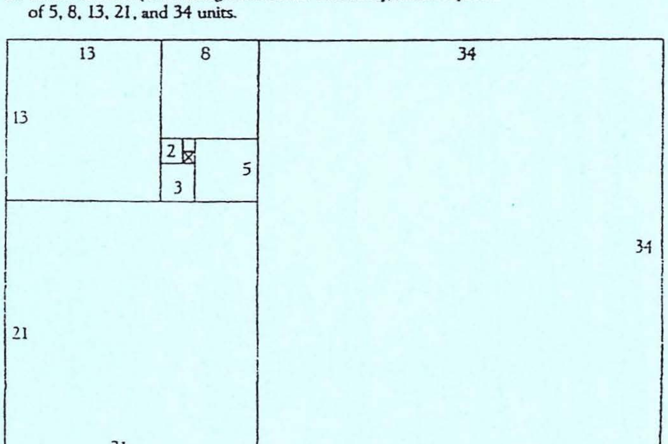
2. Attach another 1-unit square to it.

3. Attach a 2-unit square where it fits.

4. Attach a 3-unit square where it fits.



5. In like fashion (continuing in the same direction), attach squares of 5, 8, 13, 21, and 34 units.



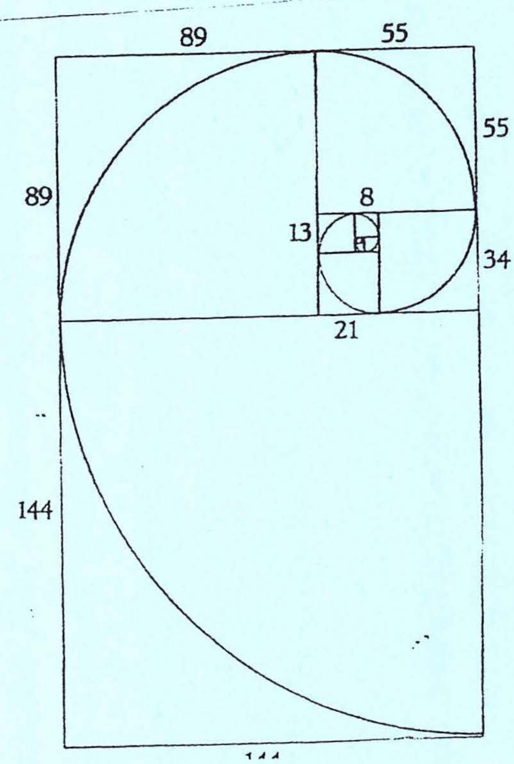
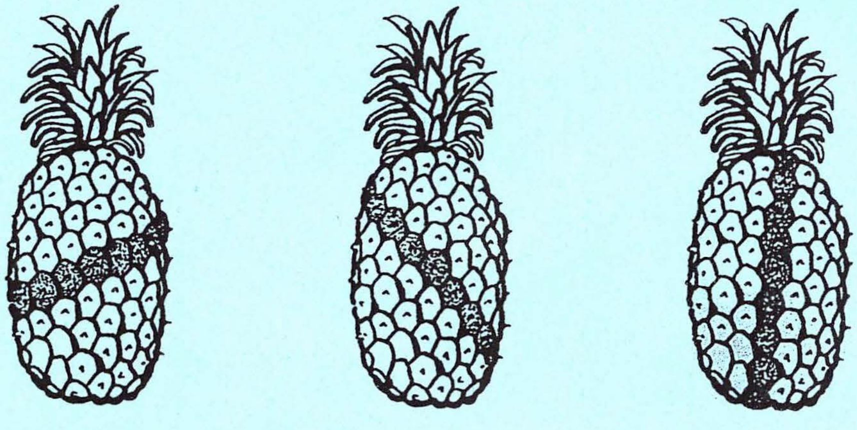
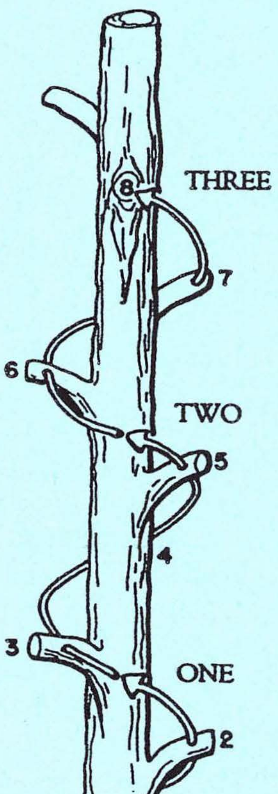


FIGURE VII

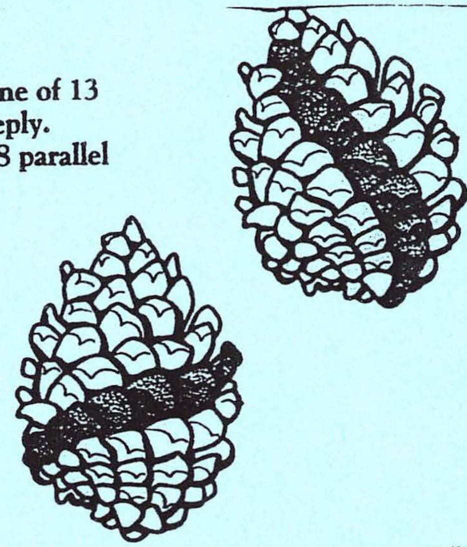


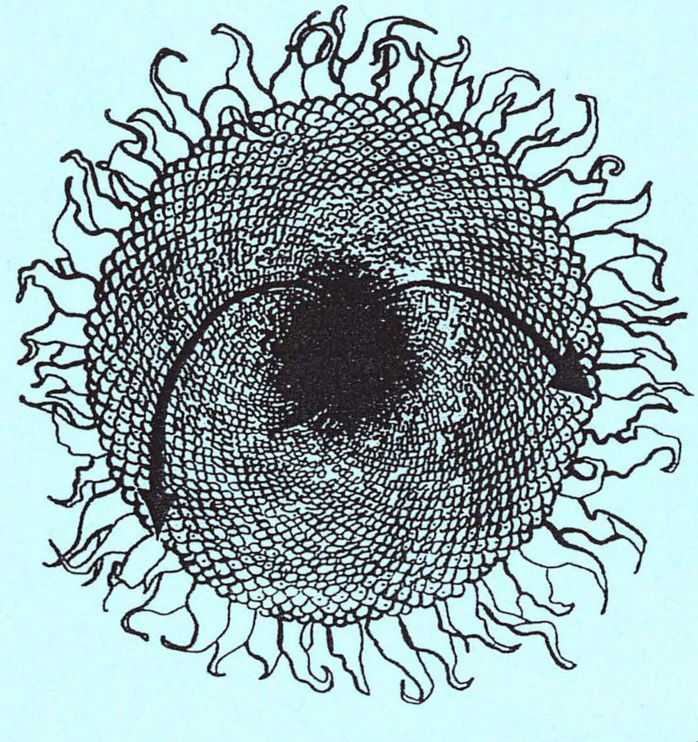
The first pineapple shows one of 8 parallel rows of scales spiraling gradually. The next pineapple shows one of 13 parallel rows of scales spiraling at a medium slope. The last pineapple shows one of 21 parallel rows of scales spiraling steeply.



Eight stems were produced in three complete growth spirals. This illustrates the phyllotactic ratio 3/8.

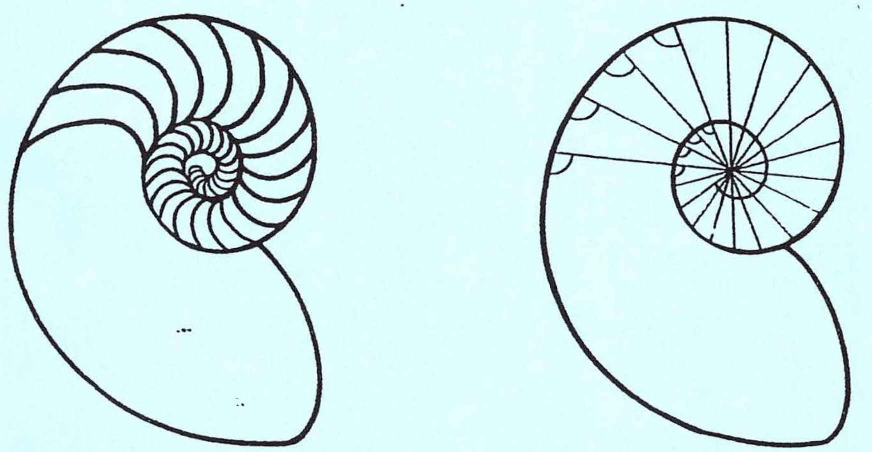
The pinecone on the right shows one of 13 parallel rows of bracts spiraling steeply. The pinecone below shows one of 8 parallel rows of bracts spiraling gradually.





Head of a sunflower showing one of 55 parallel rows of seeds spiraling counterclockwise and one of 89 parallel rows of seeds spiraling clockwise.

FIGURE VIII



On the left is a cross section of a chambered nautilus shell. On the right is a diagram of the shell showing the intersection of the radii and the shell at equal angles.

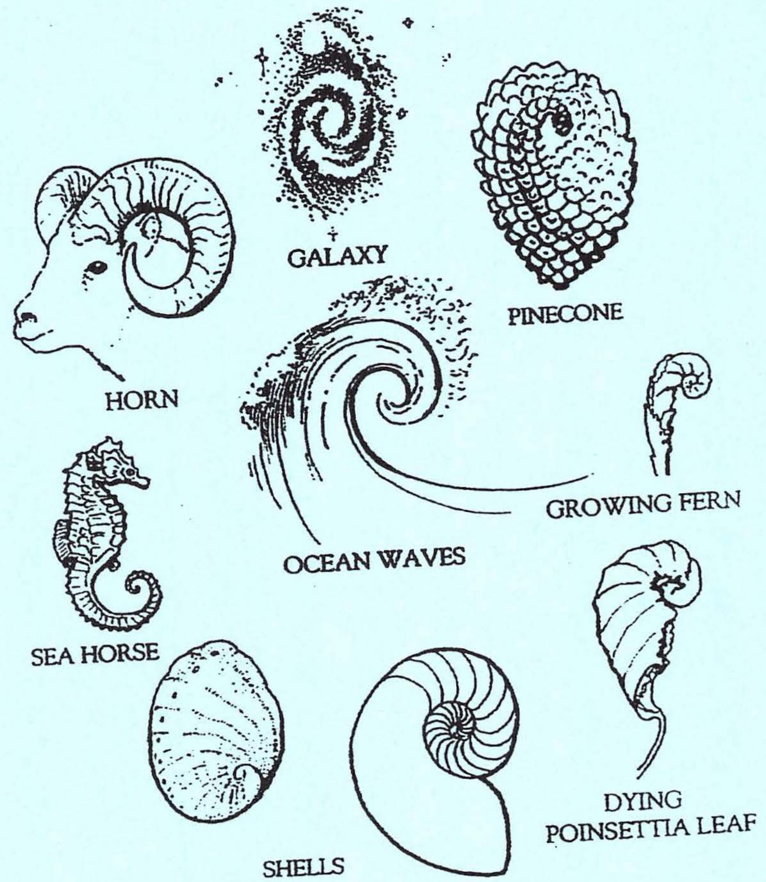


FIGURE IX

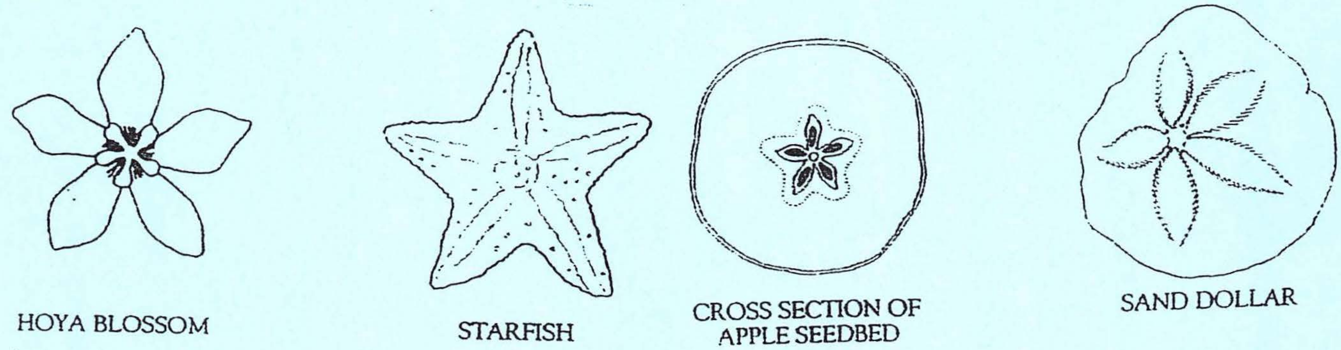
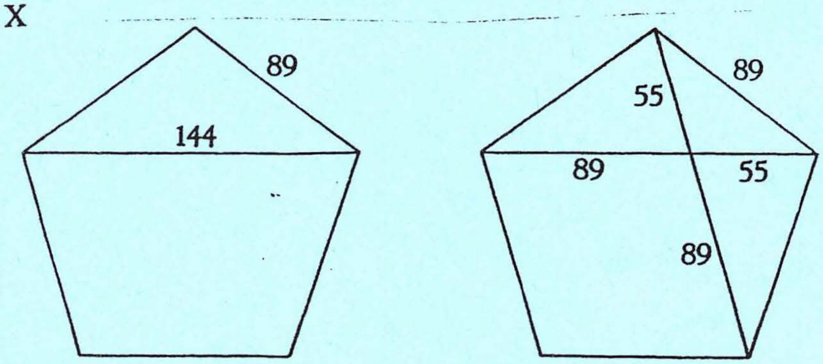


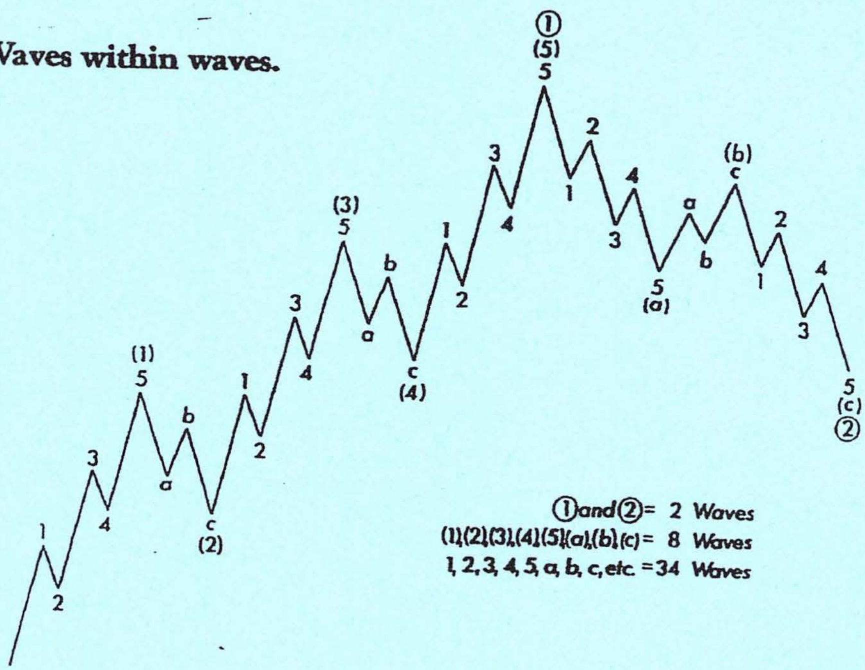
FIGURE X



The pentagon on the left shows Fibonacci proportions in the sides and diagonals. The pentagon on the right shows the diagonals intersecting in Fibonacci proportions.

FIGURE XI

Waves within waves.



Waves as parts of bigger waves.